JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 95, 20–28 (1983)

Weighted Landau Inequalities

J. A. GOLDSTEIN*

Department of Mathematics, Tulane University, New Orleans, Louisiana 70118

AND

M. K. KWONG† AND A. ZETTL†

Department of Mathematics, Northern Illinois University, De Kalb, Illinois 60115

Submitted by R. P. Boas

1. INTRODUCTION

The title refers to inequalities of the form

$$\left(\int_J |y'(t)|^p\, w(t)\, dt\right)^{2/p} \leqslant K \left(\int_J |y(t)|^p\, w(t)\, dt\right)^{1/p} \times \left(\int_J |y''(t)|^p\, w(t)\, dt\right)^{1/p}. \tag{1}$$

Here $J = \mathbb{R} = (-\infty, \infty)$ or $J = \mathbb{R}^+ = (0, \infty)$, and $1 \leqslant p < \infty$. The function w is assumed to satisfy

$$w(t) \geqslant 0 \quad \text{a.e.}, \qquad w \in L^1_{\text{loc}}(J), \qquad w(t) \neq 0 \quad \text{a.e.} \tag{2}$$

The purpose of this paper is to study the class of functions w for which (1) holds for some positive constant K and all *admissible* functions y. These are all real or complex valued functions y such that y' is absolutely continuous on compact subintervals of J and the two integrals on the right of (1) are finite. The finiteness of the integral on the left of (1) is part of the conclusion.

Clearly, if there is a positive constant K for which (1) holds for all admissible functions y, then there is a smallest such constant. This we denote by $K = K(p, J, w)$ to indicate its dependence on p, J, w.

* Partially supported by National Science Foundation Grant MCS-78-01251A01.
† Partially supported by National Science Foundation Grant MCS-78-27565.

0022-247X/83 $3.00

For a given function w, inequality (1) may not hold for any positive K as can be seen from the following simple example: $J = (0, \infty)$, $w(t) = \exp(-t)$, $y(t) = t$. On the other hand, the validity of (1) is well known and has been much studied when $w(t) = 1$ (see [6]). This leads to the question: For which functions w does there exist a positive number K such that (1) holds? This is the question we study here. We start by mentioning two known results of Kwong and Zettl [4].

THEOREM 1. *If w is nondecreasing, then* (1) *holds and*

$$K(p, \mathbb{R}, 1) \leqslant K(p, J, w) \leqslant K(p, \mathbb{R}^+, 1). \tag{3a}$$

THEOREM 2. *If $w(t) = t^c$, $c > -1$, and $J = (a, \infty)$, then* (1) *holds for some K, $0 < K < \infty$, and all $a \geqslant 0$.*

Since t^c is decreasing on $(0, \infty)$ for $-1 < c < 0$, we see that w nondecreasing is not a necessary condition for (1) to hold.

2. The Class of Weight Functions

DEFINITION. For $0 < K < \infty$, $1 \leqslant p < \infty$, let $W_p(K)$ denote the class of all real valued functions w satisfying (2) such that (1) holds for all functions y with y' absolutely continuous on compact subintervals of J for which the two integrals on the right of (1) are finite. In general $W_p(K)$ depends on K, J, and p, but it is independent of y. Let

$$W_p = \bigcup \{W_p(K): 0 < K < \infty\}. \tag{3b}$$

Finding a useful characterization of the functions in W_p or in $W_p(K)$ seems to be an exceedingly difficult problem. We establish some further sufficient conditions as well as a necessary one for a weight function to belong to W_p. By a *weight function* we mean a real valued function w satisfying conditions (2).

THEOREM 3. *The class $W_p(K)$ is a positive cone, i.e., if w_1, w_2 are in $W_p(K)$, then $a_1 w_1 + a_2 w_2$ are in $W_p(K)$ for any positive constants a_1, a_2.*

Proof. It suffices to show that $W_p(K)$ is closed under addition. Assume that (1) holds for $w = w_i$, $i = 1, 2$. Then

$$\int_J |y'|^p (w_1 + w_2) \leqslant K^{p/2} \left\{ \left(\int_J |y|^p w_1 \int_J |y''|^p w_1 \right)^{1/2} + \left(\int_J |y|^p w_2 \int_J |y''|^p w_2 \right)^{1/2} \right\}$$
$$\leqslant K^{p/2} \left(\int_J |y|^p (w_1 + w_2) \right)^{1/2} \left(\int_J |y''|^p (w_1 + w_2) \right)^{1/2},$$

where we used the Schwarz inequality for sums in the last step. ∎

It follows from Theorem 3 that W_p is also a positive cone. This result can be strengthened considerably. We assume $J = \mathbb{R}^+$ in

THEOREM 4. *The class W_p contains all positive linear combinations of functions of the form*

$$(t + a)^c\, b(t)\, w(t),$$

where $c > -1$, $a \geqslant 0$, w is a nondecreasing weight function, and b is measurable with $0 < b_0 \leqslant b(t) \leqslant B < \infty$ for some constants b_0, B and all $t \geqslant 0$.

Proof. By Theorem 3 it is enough to show that $(t + a)^c\, b(t)\, w(t)$ is in W_p. From Theorem 2 and a special case of Theorem 3 in [6] it follows that $(t + a)^c\, w(t)$ is in W_p. Thus it is sufficient to prove that w in W_p implies that bw is in W_p. To that end, let $w \in W_p$. Then,

$$\left(\int_J |y'|^p bw \right)^{2/p} \leqslant B^{2/p} \left(\int_J |y'|^p w \right)^{2/p}$$
$$\leqslant B^{2/p} K \left(\int_J |y|^p w \right)^{1/p} \left(\int_J |y''|^p w \right)^{1/p}$$
$$\leqslant KB^{2/p} \left(b_0^{-1} \int_J |y|^p bw \right)^{1/p} \left(b_0^{-1} \int_J |y''|^p bw \right)^{1/p}$$
$$= K(B/b_0)^{2/p} \left(\int_J |y|^p bw \right)^{1/p} \left(\int_J |y''|^p bw \right)^{1/p}.$$

This completes the proof of Theorem 4. ∎

Note that this proof establishes

COROLLARY 5. *Let $J = (d, \infty)$, where $-\infty \leqslant d < \infty$. If $w \in W_p(K)$, then $bw \in W_p((B/b_0)^{2/p} K)$ provided b is measurable and $0 < b_0 \leqslant b(t) < B < \infty$ for all $t \in J$.*

Theorem 2 shows that there are decreasing weight functions in W_p. However, functions in W_p cannot decrease too rapidly. Let $w(t) \leqslant ct^{-1-\varepsilon}$, $J = (1, \infty)$, and take $y(t) = t$. Then inequality (1) does not hold when $\varepsilon > p$ since the first integral on the right is finite and the second one zero while the integral on the left of (1) is positive.

COROLLARY 6. *If the positive weight function k does not decrease too rapidly in the sense that*

$$\sup_{t \in J, s \geqslant 0} \frac{k(t)}{k(t+s)} = M < \infty, \tag{4}$$

then $k \in W_p(M^{2/p} K(p, \mathbb{R}^+, 1))$ for $1 \leqslant p < \infty$.

Proof. Condition (4) implies that

$$k(t) = b(t)\, w(t)$$

for $t \in J$, where $w(t)$ is nondecreasing and $1 \leqslant b(t) \leqslant M < \infty$. Thus the conclusion of Corollary 6 follows from Corollary 5 and Theorem 1. ∎

Note that the case $M = 1$ of Corollary 6 is equivalent to k being nondecreasing and thus is contained in Theorem 1.

Using the theory of semigroups of linear operators, a different bound for K can be established.

THEOREM 7. *Suppose the weight function k satisfies* (4) *and $J = (a, \infty)$, where $-\infty \leqslant a < \infty$. Then $k \in W_p(2M^{1/p}(M^{1/p} + 1))$.*

Proof. Let $A = d/dx$ be the differentiation operator on the classical weighted Banach space $X_p = L^p(J, k)$ with the usual norm

$$\|f\|_p = \left(\int_J |f(x)|^p \, k(x)\, dx \right)^{1/p}, \qquad 1 \leqslant p < \infty.$$

It is well known that A is the infinitesimal generator of a (C_0) semigroup $\{T(t): t \geqslant 0\}$ given by

$$T(t) f(x) = f(x+t), \qquad x \in J, \quad t \geqslant 0.$$

We claim that

$$\|T(t)\| \leqslant M^{1/p}, \qquad t \geqslant 0. \tag{5}$$

This follows from

$$\| T(t)f \|_p^p = \int_J |f(t+s)|^p \, k(s) \, ds = \int_{a+t}^{\infty} |f(u)|^p \, k(u-t) \, du$$
$$\leqslant \int_{a+t}^{\infty} |f(u)|^p \, Mk(u) \, du \leqslant M \| f \|_p^p.$$

Now, (5) implies that

$$\|Af\|_p^2 \leqslant 2M^{1/p}(M^{1/p}+1) \| f \|_p \, \| A^2 f \|_p. \tag{6}$$

To see this, take the semigroup extension of Taylor's formula [2],

$$T(t)f = f + tAf + \int_0^t (t-s) \, T(s) \, A^2 f \, ds \tag{7}$$

for $f \in \operatorname{Dom}(A^2)$. Taking norms in (7) and solving for $\|Af\|$, we obtain

$$\|Af\| \leqslant t^{-1}(M^{1/p}+1) \| f \| + 2^{-1} t M^{1/p} \, \| A^2 f \|.$$

Minimizing the right side over $t > 0$ gives (6) and completes the proof of Theorem 7. ∎

Remark. If A generates a (C_0) group of isometries on a Banach space $(X, |\cdot|)$, then $|Af|^2 \leqslant 2 \, |A^2 f| \, |f|$ holds for all $f \in \operatorname{Dom}(A^2)$ by Ditzian [1]. Now let A generate a (C_0) group $\{T(t): -\infty < t < \infty\}$ on $(X, \|\cdot\|)$ satisfying $N = \sup_{t \in \mathbb{R}} \| T(t) \| < \infty$. Then $|\cdot|$ defined by

$$|f| = \sup_{t \in \mathbb{R}} \| T(t) f \|$$

is an equivalent norm on X with respect to which $\{T(t): t \in \mathbb{R}\}$ is a (C_0) group of isometries. Moreover, for $f \in \operatorname{Dom}(A^2)$,

$$\|Af\|^2 \leqslant |Af|^2 \leqslant 2 \, |A^2 f| \, |f| \leqslant 2N^2 \, \|A^2 f\| \, \|f\|.$$

Thus for $a = -\infty$ we can improve the conclusion of Theorem 7 to read $k \in W^p(2M^{1/p})$.

It is interesting to compare the bound $K = 2M^{1/p}(M^{1/p}+1)$ of Theorem 7 with the bound $K = M^{2/p} K(p, \mathbb{R}^+, 1)$ of Corollary 6. Since the exact values of $K(p, \mathbb{R}^+, 1)$ are known only for $p = 1, 2$, and ∞ [6] a precise comparison can be made only in these cases. In the other cases ($1 < p < \infty$, $p \neq 2$), the upper bound $K(p, \mathbb{R}^+, 1) \leqslant 4$ can be used. (See [5] for this and for an improved bound for certain values of p.) For $p = 2$, Corollary 6 gives a better constant K for all values of $M \geqslant 1$. Since, by [5], $\lim_{p \to 2} K(p, \mathbb{R}^+, 1) = 2$, Corollary 6 gives a better constant K for p near 2 and M bounded. When $p = 1$, $K(1, \mathbb{R}^+, 1) = \frac{5}{2}$ [6], and so Corollary 6 gives a better constant for $M < 4$, whereas Theorem 7 yields a smaller K for $M > 4$. There is agreement when $M = 4$. Since it is known [5] that $K(p, \mathbb{R}^+, 1)$ is a continuous function of p, we see that for p near 1, Corollary 6 gives a smaller constant for "small" values of $M \geqslant 1$ whereas Theorem 7 yields a smaller value of K for M large. For a discussion of the semigroup approach to the study of inequality (1) and its operator-theoretic extensions see [2].

3. Weight Functions with Zeros

The results of Section 2 do not allow a weight function w for which inequality (1) holds to have a zero at some point x_0 without being identically zero to the right or left of x_0. Here we consider weight functions w which have isolated zeros. It turns out that the validity of inequality (1) in such cases is a rather delicate matter which depends on the relationship between the order of such zeros and the value of p in (1).

Theorem 8. *Suppose w is a weight function, i.e., satisfies* (2) *on $J = (a, \infty)$, $-\infty \leqslant a < \infty$. Then w is in W_p if the following conditions are satisfied: There exist constants c_1, c_2 such that if I is any compact subinterval of J and I_i, $i = 1, 2, 3$ denote the first, second, and third thirds of I, respectively, then*

$(*)$ $\int_I w / \int_{I_i} w \leqslant c_1$, $i = 1, 3$,

$(**)$ $\int_I w^{-q/p} < \infty$ *for* $1 < p < \infty$, $p^{-1} + q^{-1} = 1$,

or $w^{-1} \in L^\infty(I)$ *if* $p = 1$,

$(***)$ $\int_I w (\int_I w^{-q/p})^{p/q} \leqslant c_2 |I|^p$ *for* $1 < p < \infty$,

or $(\int_I w) \sup_I w^{-1} \leqslant c_2 |I|$ *if* $p = 1$, *where* $|I|$ *denotes the length of* I.

Proof. The proof is based on a special case of an inequality of Kwong and Zettl [4, Theorem 2] which we state as a

Lemma. *If $w^{-q/p} \in L^1(I)$ for $1 < p < \infty$, or if $w^{-1} \in L^\infty(I)$ for $p = 1$, then*

$$\int_I w |y'|^p \leqslant B \int_I |y|^p w + A \int_I |y''|^p w \tag{8}$$

for all admissible functions y, where

$$A = 2^{p-1} \int_I w \left(\int_I w^{-q/p} \right)^{p/q} \qquad \text{if} \quad 1 < p < \infty,$$

$$A = \left(\int_I w \right) \sup_I w^{-1} \qquad \text{if} \quad p = 1,$$

$$B = 2^{2(p-1)} |I|^{-p} \int_I w \Big/ \left(\min_{i=1,3} \int_{I_i} w \right).$$

Returning to the proof of Theorem 8, let $J = \bigcup_{i=1}^{\infty} \tilde{I}_i$, where $\tilde{I}_i$ has length L and the interiors of the $\tilde{I}_i$'s are disjoint. Using hypotheses $(*)$ and $(***)$ in (8) we get

$$\int_J |y'|^p w \leqslant c_1 2^{2(p-1)} L^{-p} \int_J |y|^p w + c_2 2^{p-1} L^p \int_J |y''|^p w.$$

Minimizing the right side of this inequality over all $L > 0$ gives

$$\int_J |y'|^p w \leqslant 2(c_1 2^{2(p-1)} c_2 2^{p-1})^{1/2} \left(\int_J |y|^p w \int_J |y''|^p w \right)^{1/2}.$$

This completes the proof of Theorem 8. ∎

As we shall see, the condition $\int_I w^{-q/p} < \infty$ restricts the type of zeros a smooth function w can have. For instance, if

$$w(x) = c(x - x_0)^r$$

for $x_0 \leqslant x \leqslant x_0 + h$ with $c > 0$, $h > 0$, then with $I = [x_0, x_0 + h]$ we have

$$\int_I w^{-q/p} = c_1 \int_0^h t^{-qr/p}\, dt,$$

which is finite if and only if $-qr/p > 1$, which is equivalent to $p > r + 1$.

THEOREM 9. *Let $J = (a, \infty)$, $-\infty \leqslant a < \infty$. Suppose w has an isolated zero at x_0 of (exact) order r and is C^r in a neighborhood of x_0. Then* (1) *does not hold, that is, $w \notin W_p$ for $1 \leqslant p < r + 1$. On the other hand, there exist weight functions w having an isolated zero at x_0 of (exact) order r such that* (1) *holds, i.e., w is in W_p, for $r + 1 < p$.*

Proof. For the first assertion it is enough to construct a sequence of functions $\{y_n\}$ such that $Q(y_n) \to \infty$ as $n \to \infty$, where

$$Q(y) = \left(\int_J |y'|^p w \right)^2 \Big/ \left(\int_J |y|^p w \int_J |y''|^p w \right).$$

We may assume without loss of generality that $J=(0,\infty)$. Let

$$\begin{aligned} y_n(x) &= x_0 - x, && \text{if } 0 \leqslant x \leqslant x_0 - 1/n,\\ &= n^2(x-x_0)^3 + 2n(x-x_0)^2, && \text{if } x_0 - 1/n \leqslant x \leqslant x_0,\\ &= 0, && \text{if } x_0 \leqslant x. \end{aligned}$$

Note that $y_n(x_0)=0$, $y_n(x_0-1/n)=1/n$, and

$$\begin{aligned} y_n'(x) &= -1, && \text{if } 0 \leqslant x \leqslant x_0 - 1/n,\\ &= 3n^2(x-x_0)^2 + 4n(x-x_0), && \text{if } x_0 - 1/n \leqslant x \leqslant x_0,\\ &= 0, && \text{if } x_0 \leqslant x; \end{aligned}$$

$$\begin{aligned} y_n''(x) &= 0, && \text{if } 0 \leqslant x < x_0 - 1/n,\\ &= 6n^2(x-x_0) + 4n, && \text{if } x_0 - 1/n < x < x_0,\\ &= 0, && \text{if } x_0 < x. \end{aligned}$$

Also we have,

$$\int_J |y_n'(x)|^p\, w(x)\, dx \geqslant \int_0^{x_0-1/n} w(x)\, dx > 2^{-1} \int_0^{x_0} w(x)\, dx = K_1 > 0$$

for n large enough. Moreover, again for large n,

$$\begin{aligned} &\int_J |y_n|^p\, w\\ &\quad \leqslant \int_0^{x_0-1/n} |x-x_0|^p\, w(x)\, dx + \int_{x_0-1/n}^{x_0} (2n^{-1})^p\, w(x)\, dx\\ &\quad \leqslant \int_{1/n}^{x_0} t^p\, w(x_0-t)\, dt + 1 \leqslant \int_0^{x_0} t^p\, w(x_0-t)\, dt + 1\\ &\quad = K_0 > 0, \end{aligned}$$

$$\begin{aligned} &\int_J |y_n''|^p\, w\\ &\quad = \int_{x_0-1/n}^{x_0} [6n^2(x-x_0)+4n]^p\, w(x)\, dx\\ &\quad \leqslant 2^{p-1} \int_{x_0-1/n}^{x_0} [6^p n^{2p} |x-x_0|^p + 4^p n^p]\, w(x)\, dx = u(n,p). \end{aligned}$$

To complete the proof it suffices to show $u(n,p) \to 0$ as $n \to \infty$. Since w has a zero of order r at x_0 and is C^r in a neighborhood of x_0, it follows from Taylor's theorem that

$$w(x) = \frac{(x - x_0)^r}{r!} w^{(r)}(t_x) \leqslant C\,|x - x_0|^r$$

for $x_0 - 1/n \leqslant x \leqslant x_0$. Thus

$$\begin{aligned}
u(n,p) &\leqslant 2^{p-1} C \int_{x_0 - 1/n}^{x_0} \{6^p n^p\,|x - x_0|^p + 4^p n^p\}|x - x_0|^p\,dx \\
&= 2^{p-1} C \int_0^{1/n} (6^p n^{2p} t^{p+r} + 4^p n^p t^r)\,dt \\
&= 2^{p-1} C \{6^p n^{p-r-1}/(p + r + 1) + 4^p n^{p-r-1}/(r + 1)\} \to 0
\end{aligned}$$

as $n \to \infty$ if $p < r + 1$.

To prove the final assertion of Theorem 9, let

$$\begin{aligned}
w(x) &= 1, && \text{if } \ x \leqslant x_0 - 1, \\
&= |x - x_0|^r, && \text{if } \ x_0 - 1 \leqslant x < x_0 + 1, \\
&= 1, && \text{if } \ x \geqslant x_0 + 1.
\end{aligned}$$

Then w satisfies Theorem 8 $(*)$, and the rest of the hypotheses are also satisfied when $p > r + 1$ as can be seen from a direct computation. ∎

References

1. Z. Ditzian, Some remarks on inequalities of Landau and Kolmogorov, *Aequationes Math.* **12** (1975), 145–151.
2. H. A. Gindler and J. A. Goldstein, Dissipative operator versions of some classical inequalities, *J. Analyse Math.* **28** (1975), 213–238.
3. E. Hille and R. S. Phillips, "Functional Analysis and Semi-Groups", Amer. Math. Soc. Colloq. Publ., Vol. 31, Amer. Math. Soc., Providence, R. I., 1957.
4. M. K. Kwong and A. Zettl, Weighted norm inequalities of sum form involving derivatives, *Proc. Roy. Soc. Edinburgh* **88A** (1981), 121–134.
5. M. K. Kwong and A. Zettl, Ramifications of Landau's inequality, *Proc. Roy. Soc. Edinburgh* **86A** (1980), 175–212.
6. M. K. Kwong and A. Zettl, Norm inequalities for derivatives, *in* "Proc. Dundee Conf., 1980," Lecture Notes in Mathematics Springer-Verlag, Berlin/New York, 1980.
7. M. K. Kwong and A. Zettl, Norm inequalities of product form in weighted L^p spaces, *Proc. Roy. Soc. Edinburgh* **89A** (1981), 293–307.